

A PLETHYSM FORMULA ON THE CHARACTERISTIC MAP OF INDUCED LINEAR CHARACTERS FROM $U_n(\mathbb{F}_q)$ TO $GL_n(\mathbb{F}_q)$

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ABSTRACT. This paper gives a plethysm formula on the characteristic map of the induced linear characters from the unipotent upper-triangular matrices $U_n(\mathbb{F}_q)$ to $GL_n(\mathbb{F}_q)$, the general linear group over finite field \mathbb{F}_q . The result turns out to be a multiple of a twisted version of the Hall-Littlewood symmetric functions $\tilde{P}_n(Y, q)$. A recurrence relation is also given which makes it easy to carry out the computation.

1. INTRODUCTION

Let \mathbb{F}_q be a fixed finite field and $GL_n(\mathbb{F}_q)$ the finite general linear group over \mathbb{F}_q . The representation theory of $GL_n(\mathbb{F}_q)$ over \mathbb{C} has been thoroughly studied by J.A.Green [4]. He also constructed the characteristic map which builds a connection between the character spaces of $GL_n(\mathbb{F}_q)$ for $n \geq 0$ and the Cartesian product over infinitely indexed sets of rings of symmetric functions. In character theory, the study of induced linear characters from subgroups is very useful to understand the character ring of the larger group. In this paper, we consider certain induced linear characters from the group of unipotent upper-triangular matrices $U_n(\mathbb{F}_q)$ to $GL_n(\mathbb{F}_q)$. The representations of these induced linear characters are known as Gelfand-Graev modules, which play an important role in the representation theory of finite groups of Lie type ([3], [10]). The formula for the characteristic map of the induced linear characters is given by Thiem [7]. We then apply plethysms on the image of the characteristic map. There are two advantages in doing so: to get a simpler formula and to express the result as a multiple of a twisted version of the Hall-Littlewood symmetric functions $\tilde{P}_n(y, q)$. We hope this method could contribute to the study on the irreducible decomposition of the induced characters from $U_n(\mathbb{F}_q)$ to $GL_n(\mathbb{F}_q)$.

In section 2 we give some background knowledge on symmetric functions and representation theory of $GL_n(\mathbb{F}_q)$ and $U_n(\mathbb{F}_q)$. Since the character theory of $U_n(\mathbb{F}_q)$ is known as a wild problem, supercharacter theory is built up as an approximation of the ordinary character theory. The linear characters of $U_n(\mathbb{F}_q)$ that we are considering are part of the category of supercharacters of $U_n(\mathbb{F}_q)$. We introduce further questions about the induction of all supercharacters in Section 4. In Section 3 we give our main result about the plethysm formula. A recurrence relation is obtained naturally so that we can carry out the computation of plethysms on the characteristic map of the induced linear characters more easily. We also give a relation between the characteristic map of the induced characters from $U_n(\mathbb{F}_q)$ to $GL_n(\mathbb{F}_q)$ and the plethysms

on those characteristics. This is depicted in the following diagram

$$\begin{array}{ccc} \otimes_{\varphi \in \Theta} \Lambda_{\mathbb{C}}(Y^{\varphi}) & \xrightarrow{T} & \otimes_{f \in \Phi} \Lambda_{\mathbb{C}}(X_f) \\ \rho \downarrow & & \downarrow \Pi|_{\Lambda_{\mathbb{C}}(X_{f=x-1})} \\ \Lambda_{\mathbb{C}}(Y) & \xrightarrow{t \circ \omega} & \Lambda_{\mathbb{C}}(X_{x-1}) \end{array}$$

where the notation is explained in Theorem 3.16. From the above commutative diagram we show that our simplified plethysm formula does not lose any information on the characteristic map of the induced characters from $U_n(\mathbb{F}_q)$ to $GL_n(\mathbb{F}_q)$.

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2. BACKGROUND

2.1. Symmetric functions. The notation in this paper follows closely the book of Macdonald [6].

Definition 2.1. A *partition* λ of $n \in \mathbb{N}$, is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of positive integers in weakly decreasing order: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l$, such that $\lambda_1 + \lambda_2 + \dots + \lambda_l = n$. We denote this by $\lambda \vdash n$. Here, each λ_i ($1 \leq i \leq l$) is called a *part* of λ . We say the *length* of the partition λ is l , which is the number of parts of λ . We use $|\lambda|$ to denote the sum of all parts $\lambda_1 + \lambda_2 + \dots + \lambda_l$, and we call $|\lambda|$ the *size* of the partition. Sometimes we also use the notation:

$$\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n}, \dots)$$

where each m_i means there are m_i parts in λ equal to i .

Let $\Lambda_{\mathbb{C}}(Y)$ denote the ring of symmetric functions with complex coefficients in the variables $Y = \{y_1, y_2, \dots\}$. We denote the complete symmetric functions, elementary symmetric functions, monomial symmetric functions, power-sum symmetric functions, and Schur symmetric functions by $h_{\lambda}(Y)$, $e_{\lambda}(Y)$, $m_{\lambda}(Y)$, $p_{\lambda}(Y)$, and $s_{\lambda}(Y)$ respectively.

The generating function for $h_n(Y)$ is

$$H(Y; t) = \sum_{n \geq 0} h_n(Y) t^n = \prod_{j \geq 1} (1 - y_j t)^{-1}.$$

Let $X = \{x_1, x_2, \dots\}$ be another set of finite or infinite variables. We have the following identity:

$$(2.1) \quad \Omega[XY] := \prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} m_{\lambda}(X) h_{\lambda}(Y)$$

summed over all partitions λ .

There is a scalar product defined on $\Lambda_{\mathbb{C}}(Y)$, which makes (m_{λ}) and (h_{λ}) dual to each other:

$$\langle h_{\lambda}, m_{\mu} \rangle = \delta_{\lambda\mu}$$

for all partitions λ, μ , where $\delta_{\lambda\mu}$ is the Kronecker delta.

We use $P_\lambda(Y; t)$ to denote the Hall-Littlewood symmetric functions as defined in [6]. If we define

$$\begin{aligned} q_r &= q_r(Y; t) = (1 - t)P_{(r)}(Y; t) \quad \text{for } r \geq 1, \\ q_0 &= q_0(Y; t) = 1, \end{aligned}$$

then the generating function for $q_r(Y; t)$ is

$$(2.2) \quad Q(u) = \sum_{r \geq 0} q_r(Y; t) u^r = \prod_i \frac{1 - y_i t u}{1 - y_i u}.$$

For each partition λ let $n(\lambda) = \sum_{i \geq 1} (i - 1)\lambda_i$. Define

$$\tilde{P}_\lambda(Y; q) = q^{-n(\lambda)} P_\lambda(Y; q^{-1})$$

and we call $\tilde{P}_\lambda(Y; q)$ the twisted Hall-Littlewood symmetric functions.

From [6] it is well known that the *plethysm* can be defined by

$$(2.3) \quad h_a[p_b] = h_a(y_1^b, y_2^b, \dots),$$

which is the coefficient of t^{ab} in $\prod_{j \geq 1} (1 - y_j^b t^b)^{-1}$.

2.2. Representation theory of $GL_n(\mathbb{F}_q)$. The representation theory of the finite general linear group $G_n = GL_n(\mathbb{F}_q)$ over \mathbb{C} can be found in J.A.Green [4], Macdonald [6] and Thiem [7]. Here we give a short description on the characteristic map constructed by J.A.Green.

Let $\bar{\mathbb{F}}_q$ denote the algebraic closure of the finite field \mathbb{F}_q . The multiplicative group of $\bar{\mathbb{F}}_q$ is denoted by $\bar{\mathbb{F}}_q^\times$. Let $\bar{\mathbb{F}}_q^* = \{\phi : \bar{\mathbb{F}}_q^\times \rightarrow \mathbb{C}^\times\}$ be the group of complex-valued multiplicative characters of $\bar{\mathbb{F}}_q^\times$. The Frobenius automorphism of $\bar{\mathbb{F}}_q$ over \mathbb{F}_q is given by

$$F : x \rightarrow x^q, \text{ where } x \in \bar{\mathbb{F}}_q.$$

We then define

$$\Phi = \{F\text{-orbits of } \bar{\mathbb{F}}_q^\times\} \text{ and } \Theta = \{F\text{-orbits of } \bar{\mathbb{F}}_q^*\}.$$

Since every F -orbits of $\bar{\mathbb{F}}_q^\times$ is in one-to-one correspondence with irreducible polynomial f over \mathbb{F}_q , we can also use f to denote each F -orbit in Φ . A partition-valued function μ on Φ is a function which maps each $f \in \Phi$ to a partition $\mu(f)$. The size of μ is

$$\|\mu\| = \sum_{f \in \Phi} d(f) |\mu(f)|,$$

where $d(f)$ is equal to the degree of $f \in \Phi$.

Let \mathbb{P} denote the set of all partitions and

$$\mathcal{P}^\Phi = \bigcup_{n \geq 0} \mathcal{P}_n^\Phi, \text{ where } \mathcal{P}_n^\Phi = \{\mu : \Phi \rightarrow \mathbb{P}; \|\mu\| = n\}.$$

We use K^μ to denote the conjugacy classes in G_n parameterized by $\mu \in \mathcal{P}_n^\Phi$ [6]. The characteristic function of the conjugacy class K^μ is denoted by π_μ .

Similarly, for each partition-valued function $\lambda : \Theta \rightarrow \mathbb{P}$, the size of λ is

$$\|\lambda\| = \sum_{\varphi \in \Theta} d(\varphi) |\lambda(\varphi)|,$$

where $d(\varphi)$ is equal to the number of elements in φ . Let

$$\mathcal{P}^\Theta = \bigcup_{n \geq 0} \mathcal{P}_n^\Theta, \text{ where } \mathcal{P}_n^\Theta = \{\lambda : \Theta \rightarrow \mathbb{P}; \|\lambda\| = n\}.$$

We use G_n^λ to denote the irreducible G_n -modules indexed by $\lambda \in \mathcal{P}_n^\Theta$ [6]. The character of the irreducible G_n -modules G_n^λ is denoted by χ^λ .

For every $f \in \Phi$, let $X_f := \{X_{1,f}, X_{2,f}, \dots\}$ be a set of infinitely many variables. Each $X_{i,f}$ has degree $d(f)$.

Let

$$\tilde{P}_\eta(f) = \tilde{P}_\eta(X_f; q^{d(f)}) = q^{-d(f)n(\eta)} P_\eta(X_f; q^{-d(f)})$$

where $\tilde{P}_\eta(X_f; q^{d(f)})$ is the twisted Hall-Littlewood symmetric function. Define

$$\tilde{P}_\mu = \prod_{f \in \Phi} \tilde{P}_{\mu(f)}(f).$$

For every $\varphi \in \Theta$, let $Y^\varphi := \{Y_1^\varphi, Y_2^\varphi, \dots\}$ be a set of infinitely many variables. Each Y_i^φ has degree $d(\varphi)$. Define

$$S_\lambda = \prod_{\varphi \in \Theta} s_{\lambda(\varphi)}(Y^\varphi),$$

where $s_{\lambda(\varphi)}(Y^\varphi)$ is the Schur symmetric function.

Let

$$\Lambda_{\mathbb{C}} = \otimes_{f \in \Phi} \Lambda_{\mathbb{C}}(X_f) = \otimes_{\varphi \in \Theta} \Lambda_{\mathbb{C}}(Y^\varphi)$$

where $\Lambda_{\mathbb{C}}(X_f)$ is the ring of symmetric functions in X_f , and $\Lambda_{\mathbb{C}}(Y^\varphi)$ is the ring of symmetric functions in Y^φ . As a graded ring, we have

$$\begin{aligned} \Lambda_{\mathbb{C}} &= \mathbb{C}\text{-span}\{\tilde{P}_\mu | \mu \in \mathcal{P}^\Phi\} \\ &= \mathbb{C}\text{-span}\{S_\lambda | \lambda \in \mathcal{P}^\Theta\} \end{aligned}$$

The transformation between the symmetric functions in the variables $\{X_f : f \in \Phi\}$ and those in the variables $\{Y^\varphi : \varphi \in \Theta\}$ is given by the following identity:

$$(2.4) \quad p_k(Y^\varphi) = (-1)^{n-1} \sum_{x \in M_n} \xi(x) p_{n/d(f_x)}(X_{f_x}),$$

where $\xi \in \varphi$, $x \in f_x$ and $n = k \cdot d(\varphi)$. Here $p_k(Y^\varphi)$ and $p_{n/d(f_x)}(X_{f_x})$ are power-sum symmetric functions.

From [6] we know that the conjugacy classes K^μ of G_n are parameterized by $\mu \in \mathcal{P}_n^\Phi$, and the irreducible characters χ^λ of G_n are indexed by $\lambda \in \mathcal{P}_n^\Theta$. The following theorem gives the characteristic map of G_n .

Theorem 2.2. (Green [7], Macdonald [6], Zelevinski [12]) *Let A_n denote the space of complex-valued class functions on G_n and $A = \bigoplus_{n \geq 0} A_n$. The linear map*

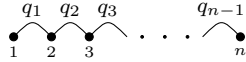
$$\begin{aligned} ch : A &\longrightarrow \Lambda_{\mathbb{C}} \\ \chi^{\lambda} &\mapsto S_{\lambda}, \quad \text{for } \lambda \in \mathcal{P}^{\Theta}, \\ \pi_{\mu} &\mapsto \tilde{P}_{\mu}, \quad \text{for } \mu \in \mathcal{P}^{\Phi}, \end{aligned}$$

is a Hopf algebra isomorphism.

2.3. Supercharacter theory. Let U_n be the group of unipotent upper-triangular matrices with entries in the finite field \mathbb{F}_q and ones on the diagonal. This group is the subgroup of the finite general linear group G_n . Although the character theory on U_n is a wild problem, another slightly coarse version called superclass and supercharacter theory (André [1], Yan [11]) makes it easier to study and compute. Superclasses are certain unions of conjugacy classes and supercharacters are sums of irreducible characters. They are compatible in the sense that supercharacters are constant on superclasses. The supercharacter theory has a rich combinatorial structure (ref. [8]) and connects to some other algebra structures as well (ref. [2]).

The superclasses of U_n can be indexed by the \mathbb{F}_q^{\times} -labeled set partitions, and a supercharacter becomes an irreducible character if the corresponding indexed \mathbb{F}_q^{\times} -labeled set partition has no crossing arcs. For the strict definitions and more details on supercharacters please see [8] or [2].

In this paper we consider the linear supercharacters of U_n indexed by



where $q_1, \dots, q_{n-1} \in \mathbb{F}_q^{\times}$ (Thiem [8]). Let $\chi_{(q_1, \dots, q_{n-1})}^{(n)}$ denote the above character. We induce $\chi_{(q_1, \dots, q_{n-1})}^{(n)}$ from U_n to G_n by the formula

$$(2.5) \quad \chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n} (g) = \frac{1}{|U_n|} \sum_{h \in G_n} \bar{\chi}_{(q_1, \dots, q_{n-1})}^{(n)} (hgh^{-1}),$$

where $\bar{\chi}(s) = \chi(s)$ if $s \in U_n$ and $\bar{\chi}(s) = 0$ if $s \notin U_n$.

The induced character $\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}$ is a character of G_n , which is known as the character of Gelfand-Graev module. Apply plethysms on the characteristic map of $\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}$ and we get a multiple of the twisted Hall-Littlewood symmetric function \tilde{P}_n . A formula on this result together with a recurrence relation is given in Section 3.

3. PLETHYSM FORMULA FOR THE INDUCED CHARACTER

We start from the formula of the characteristic map of $\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}$, which is given by Thiem [7].

Theorem 3.1. (Thiem [7])

$$(3.1) \quad ch(\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}) = \sum_{\substack{\lambda \in \mathcal{P}_n^\Theta \\ ht(\lambda)=1}} S_\lambda,$$

where $ht(\lambda) = \max\{l(\lambda(\varphi)) | \varphi \in \Theta\}$.

Notice that $ht(\lambda) = 1$ implies for every $\varphi \in \Theta$ we have $l(\lambda(\varphi)) \leq 1$, which means $\lambda(\varphi)$ contains at most one part. From the definition of S_λ , we can write (3.1) as

$$(3.2) \quad \begin{aligned} ch(\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}) &= \sum_{\substack{\lambda \in \mathcal{P}_n^\Theta \\ ht(\lambda)=1}} \prod_{\varphi \in \Theta} s_{\lambda(\varphi)}(Y^\varphi) \\ &= \sum_{\substack{a_1 b_1 + \dots + a_k b_k = n \\ \lambda(\Theta) = \{a_1, \dots, a_k\} \in \mathcal{P}_n^\Theta}} \sum_{\substack{deg(\varphi_i) = b_i \\ \varphi_1, \dots, \varphi_k \text{ distinct}}} h_{a_1}(Y^{\varphi_1}) h_{a_2}(Y^{\varphi_2}) \dots h_{a_k}(Y^{\varphi_k}), \end{aligned}$$

where $Y^{\varphi_1}, Y^{\varphi_2}, \dots, Y^{\varphi_k}$ are different sets of variables. For i from 1 to k , each variable in the set $Y^{\varphi_i} = \{Y_1^{\varphi_i}, Y_2^{\varphi_i}, \dots\}$ has degree b_i .

We give an example to better understand formula (3.1) and (3.2).

Example 3.2. For $n = 3$, we have

$$\begin{aligned} ch(\chi_{(q_1, q_2)}^{(3)} \uparrow_{U_3}^{G_3}) &= \sum_{\substack{\varphi_1, \varphi_2, \varphi_3 \text{ distinct} \\ deg(\varphi_i)=1}} h_1(Y^{\varphi_1}) h_1(Y^{\varphi_2}) h_1(Y^{\varphi_3}) \\ &+ \sum_{\substack{\psi_1, \psi_2 \text{ distinct} \\ deg(\psi_i)=1}} h_2(Y^{\psi_1}) h_1(Y^{\psi_2}) + \sum_{\substack{deg(\bar{\varphi}_1)=2 \\ deg(\bar{\varphi}_2)=1}} h_1(Y^{\bar{\varphi}_1}) h_1(Y^{\bar{\varphi}_2}) \\ &+ \sum_{deg(\varphi)=1} h_3(Y^\varphi) + \sum_{deg(\psi)=3} h_1(Y^\psi). \end{aligned}$$

From the above example we see that the expansion on the right-hand side of (3.2) becomes more complicated as n increases. This inspires us to use plethysm to simplify the computation.

For each term in equation (3.2), we have a *two-rowed* array $\begin{pmatrix} b_1 & b_2 & \dots & b_k \\ a_1 & a_2 & \dots & a_k \end{pmatrix}$ where $b_i = d(\varphi_i)$ and it satisfies the condition $\sum_{i=1}^k a_i b_i = n$. We arrange the pairs (b_i, a_i) such that:

- (1) $b_1 \leq b_2 \leq \dots \leq b_k$,
- (2) $a_j \leq a_{j+1}$ if $b_j = b_{j+1}$ for $1 \leq j < k$.

Once the array is sorted, we can denote it as follows:

$$\left(\begin{array}{cccccccccccc} 1^{m_1} & & & & 2^{m_2} & & \dots & & n^{m_n} & & & \\ 1^{m_{1,1}} & 2^{m_{1,2}} & \dots & n^{m_{1,n}} & 1^{m_{2,1}} & 2^{m_{2,2}} & \dots & n^{m_{2,n}} & \dots & 1^{m_{n,1}} & 2^{m_{n,2}} & \dots & n^{m_{n,n}} \end{array} \right)$$

where $\sum_{i,j=1}^n m_{i,j} \times j \times i = n$ and $m_{i,1} + m_{i,2} + \dots + m_{i,n} = m_i$ for $1 \leq i \leq n$. Each m_i counts the number of different sets of variables appearing in the term with the same degree i . Each $m_{i,j}$ counts the number of complete symmetric functions h_j in variables with degree i .

For a given i , let $l_q(i)$ denote the number of all different sets of variables with the same degree i . We know that $l_q(i)$ is equal to the number of irreducible polynomials f over finite field \mathbb{F}_q with degree i and satisfying $f(0) \neq 0$. The number of irreducible polynomials of degree i over \mathbb{F}_q is given by the fomula

$$L_q(i) = \frac{1}{i} \sum_{d|i} \mu(d) q^{\frac{i}{d}},$$

where μ is the Möbius function. Then we have

$$l_q(i) = \begin{cases} L_q(1) - 1, & \text{for } i = 1; \\ L_q(i), & \text{for } i \geq 2. \end{cases}$$

Thus for a given i and a list of numbers $(m_{i,1}, m_{i,2}, \dots, m_{i,n})$ where $m_{i,1} + m_{i,2} + \dots + m_{i,n} = m_i$, the number of products in the form

$$\begin{aligned} & h_1(Y^{\varphi_{i,1}}) h_1(Y^{\varphi_{i,2}}) \dots h_1(Y^{\varphi_{i,m_{i,1}}}) \\ & \times h_2(Y^{\varphi_{i,m_{i,1}+1}}) h_2(Y^{\varphi_{i,m_{i,1}+2}}) \dots h_2(Y^{\varphi_{i,m_{i,1}+m_{i,2}}}) \\ & \times \dots \\ (3.3) \quad & \times h_n(Y^{\varphi_{i,m_{i,1}+\dots+m_{i,n-1}+1}}) h_n(Y^{\varphi_{i,m_{i,1}+\dots+m_{i,n-1}+2}}) \dots h_n(Y^{\varphi_{i,m_i}}) \end{aligned}$$

is equal to

$$\frac{l_q(i)(l_q(i) - 1) \dots (l_q(i) - m_i + 1)}{m_{i,1}! m_{i,2}! \dots m_{i,n}!},$$

where $Y^{\varphi_{i,1}}, Y^{\varphi_{i,2}}, \dots, Y^{\varphi_{i,m_i}}$ are m_i different sets of variables with the same degree i . Notice that when n increases, we get more terms on the right-hand side of equation (3.2).

In order to simplify the computation, we apply plethysms on (3.2) which means replacing each set of variables Y^{φ_i} by $\{y_1^{b_i}, y_2^{b_i}, \dots\}$. In doing so we don't differentiate the sets of variables. It seems that we lose information by applying the plethysms on the characteristic map. However this is not the case as we see later on in Theorem 3.16 and Corollary 3.17.

Definition 3.3. Define the *plethysm map* $\rho : \mathbb{C}\text{-span}\{S_{\lambda} | \lambda \in \mathcal{P}^{\Theta}\} \rightarrow \Lambda_{\mathbb{C}}(Y)$ as follows:

$$\rho(h_a(Y^{\varphi})) = h_a[p_b(Y)], \quad \forall \varphi \in \Theta, b = \deg(\varphi).$$

Since $ch(\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n})$ is independent from q_1, \dots, q_{n-1} , for $n \geq 1$ we simply denote $\rho(ch(\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}))$ by ρ_n and set $\rho_0 = 1$. We also use $\rho_{(m_{i,1}, \dots, m_{i,n})}$ to denote the results of taking plethysms on the sum of all different products in the form of (3.3) for the same index list $(m_{i,1}, \dots, m_{i,n})$, i.e.

$$\rho_{(m_{i,1}, \dots, m_{i,n})} := \frac{l_q(i)(l_q(i) - 1) \dots (l_q(i) - m_i + 1)}{m_{i,1}! m_{i,2}! \dots m_{i,n}!} (h_1[p_i])^{m_{i,1}} \dots (h_n[p_i])^{m_{i,n}}.$$

Taking plethysms on both sides of (3.2) we get

$$(3.4) \quad \rho_n = \sum_{\sum_{i,j=1}^n m_{i,j} \times j = n} \rho_{(m_{1,1}, \dots, m_{1,n})} \dots \rho_{(m_{n,1}, \dots, m_{n,n})}.$$

The following theorem falls naturally.

Theorem 3.4. Let $CH(t)$ denote the generating function for ρ_n as follows:

$$CH(t) = 1 + \rho_1 t + \rho_2 t^2 + \cdots = \sum_{s \geq 0} \rho_s t^s.$$

Then we have

$$CH(t) = \prod_{i \geq 1} \left(\prod_{j \geq 1} (1 - y_j^i t^i)^{-1} \right)^{Lq(i)} = \prod_{i \geq 1} \prod_{j \geq 1} (1 - y_j^i t^i)^{-Lq(i)}.$$

Proof. Since for every $i \geq 1$,

$$\begin{aligned} \prod_{j \geq 1} (1 - y_j^i t^i)^{-1} &= \sum_{a \geq 0} h_a(y_1^i, y_2^i, \dots) t^{a \cdot i} \\ &= 1 + (h_1[p_i]) \cdot t^i + (h_2[p_i]) \cdot t^{2i} + \cdots. \end{aligned}$$

We have

$$\begin{aligned} &\left(\prod_{j \geq 1} (1 - y_j^i t^i)^{-1} \right)^{Lq(i)} \\ &= \left(1 + (h_1[p_i]) t^i + (h_2[p_i]) t^{2i} + \cdots \right)^{Lq(i)} \\ &= \sum_{\substack{m_{i,1} + m_{i,2} + \cdots + m_{i,n} = m_i \\ 0 \leq m_i \leq Lq(i)}} \binom{Lq(i)}{m_i} \binom{m_i}{m_{i,1} m_{i,2} \cdots m_{i,n}} \\ &\quad \times (h_1[p_i])^{m_{i,1}} (h_2[p_i])^{m_{i,2}} \cdots (h_n[p_i])^{m_{i,n}} \cdot t^{(m_{i,1} + 2m_{i,2} + \cdots + n \cdot m_{i,n}) \cdot i} \\ &= \sum_{\substack{m_{i,1} + m_{i,2} + \cdots + m_{i,n} = m_i \\ 0 \leq m_i \leq Lq(i)}} \rho_{(m_{i,1}, \dots, m_{i,n})} \cdot t^{(m_{i,1} + 2m_{i,2} + \cdots + n \cdot m_{i,n}) \cdot i} \end{aligned}$$

From (3.4) we see that the coefficient of t^n in the product $\prod_{i \geq 1} \left(\prod_{j \geq 1} (1 - y_j^i t^i)^{-1} \right)^{Lq(i)}$ is exactly equal to ρ_n for $n \geq 1$. Thus we get the Theorem. \square

Theorem 3.5.

$$(3.5) \quad \prod_{i \geq 1} \prod_{j \geq 1} (1 - y_j^i t^i)^{-Lq(i)} = \frac{\prod_{j \geq 1} (1 - y_j q t)^{-1}}{\prod_{j \geq 1} (1 - y_j t)^{-1}}.$$

Proof. The above identity is equivalent to the identity

$$(3.6) \quad \prod_{i \geq 1} \prod_{j \geq 1} (1 - y_j^i t^i)^{Lq(i)} = \prod_{j \geq 1} (1 - y_j q t),$$

where $Lq(i)$ denotes the number of irreducible polynomials over \mathbb{F}_q for $i \geq 1$ as we stated before. To prove (3.6), we take the logarithm on both sides of (3.6) and show

they are equal.

$$\begin{aligned}
\ln \left(\prod_{i \geq 1} \prod_{j \geq 1} (1 - y_j^i t^i)^{Lq(i)} \right) &= \sum_{j \geq 1} \left(\sum_{i \geq 1} Lq(i) \ln(1 - y_j^i t^i) \right) \\
&= \sum_{j \geq 1} \left(\sum_{i \geq 1} Lq(i) \left(\sum_{r \geq 1} \frac{(y_j^i t^i)^r}{r} \right) \right) \\
&= \sum_{j \geq 1} \left(\sum_{i \geq 1} \sum_{r \geq 1} Lq \left(\frac{i \cdot r}{r} \right) \cdot \frac{i \cdot r}{r} \cdot \frac{y_j^{(i \cdot r)} \cdot t^{(i \cdot r)}}{i \cdot r} \right) \\
&= \sum_{j \geq 1} \left(\sum_{\substack{N \geq 1 \\ N = i \cdot r}} \frac{y_j^{(N)} \cdot t^{(N)}}{N} \left(\sum_{r|N} Lq \left(\frac{N}{r} \right) \cdot \frac{N}{r} \right) \right) \\
&= \sum_{j \geq 1} \left(\sum_{\substack{N \geq 1 \\ N = i \cdot r}} \frac{y_j^{(N)} \cdot t^{(N)}}{N} \cdot q^N \right) \\
&= \sum_{j \geq 1} (\ln(1 - y_j q t)) = \ln \left(\prod_{j \geq 1} (1 - y_j q t) \right)
\end{aligned}$$

Thus we get (3.6). □

Theorem (3.4) and Theorem (3.5) together yield the formula for the generating function of ρ_n as follows:

$$(3.7) \quad CH(t) = \prod_{j \geq 1} \frac{1 - y_j t}{1 - y_j q t}.$$

Before we link it to Hall-Littlewood polynomials, we give a recurrence relation for ρ_n using formula (3.7).

Corollary 3.6. *For every $n \geq 1$, we have*

$$(3.8) \quad \rho_n = (q^n - 1)h_n - \rho_{n-1}h_1 - \rho_{n-2}h_2 \cdots - \rho_1 h_{n-1}.$$

Proof. From (3.7) we have

$$CH(t) \times H(t) = \prod_{j \geq 1} (1 - y_j q t)^{-1}.$$

Compare the coefficients of t^n on both sides we get

$$\rho_0 h_n + \rho_1 h_{n-1} + \cdots + \rho_n h_0 = q^n h_n,$$

which yields the theorem. □

Example 3.7.

$$\begin{aligned}
\rho_1 &= (q-1)h_1; \\
\rho_2 &= (q^2-1)h_2 - \rho_1 h_1 \\
&= (q^2-1)h_2 - (q-1)h_{1,1} \\
&= (q-1)[(q+1)h_2 - h_{1,1}]; \\
\rho_3 &= (q^3-1)h_3 - \rho_1 h_2 - \rho_2 h_1 \\
&= (q^3-1)h_3 - (q-1)h_{2,1} - (q^2-1)h_{2,1} + (q-1)h_{1,1,1} \\
&= (q-1)[(q^2+q+1)h_3 - (q+2)h_{2,1} + h_{1,1,1}].
\end{aligned}$$

From the above examples we notice that the coefficients of h_λ are in $\pm\mathbb{N}[q] \times (q-1)$. Let $[h_\lambda]\rho_n$ denote the coefficients of h_λ in the expansion of ρ_n . In particular we have $[h_n]\rho_n = q^n - 1$ for all $n \geq 1$. The following corollary gives the recurrence relation on the coefficients.

Corollary 3.8. *For any $\lambda = (a_1^{l_1}, a_2^{l_2}, \dots, a_k^{l_k}) \vdash n$ with $l_i \geq 1$ for all $1 \leq i \leq k$ and $l(\lambda) \geq 2$, we have*

$$(3.9) \quad [h_\lambda]\rho_n = -[h_{(a_1^{l_1-1}, a_2^{l_2}, \dots, a_k^{l_k})}]\rho_{n-a_1} - \dots - [h_{(a_1^{l_1}, a_2^{l_2}, \dots, a_k^{l_k-1})}]\rho_{n-a_k}.$$

Here if $l_i = 1$ for some $1 \leq i \leq k$, then we set

$$(a_1^{l_1}, \dots, a_i^{l_i-1}, \dots, a_k^{l_k}) := (a_1^{l_1}, \dots, \hat{a}_i, \dots, a_k^{l_k}),$$

where \hat{a}_i means simply remove a_i from the partition λ . In particular, $[h_\lambda]\rho_n \in \pm\mathbb{N}[q] \times (q-1)$ while the sign is given by $(-1)^{l(\lambda)-1}$.

Proof. Equation (3.9) follows directly from Corollary 3.6 by comparing the coefficients of h_λ from two sides. The claim that $[h_\lambda]\rho_n$ is in $\pm\mathbb{N}[q] \times (q-1)$ together with the sign property can be proved easily by using induction method on equation (3.9). \square

Remark 3.9. Corollary 3.6 and Corollary 3.8 give an easy way of computing ρ_n for every $n \geq 1$ simply by knowing $[h_i]\rho_i = q^i - 1$ for every $i \geq 1$.

Example 3.10.

$$\begin{aligned}
[h_{2,1}]\rho_3 &= -[h_1]\rho_1 - [h_2]\rho_2 \\
&= -(q-1) - (q^2-1) \\
&= -(q-1)(q+2)
\end{aligned}$$

$$\begin{aligned}
[h_{1,1,1}]\rho_3 &= -[h_{1,1}]\rho_2 = [h_1]\rho_1 \\
&= q-1
\end{aligned}$$

Now back to our formula (3.7). We rewrite it into the following form so that we can easily use the generating function for q_r as in equation (2.2).

$$\begin{aligned} CH(t) &= \prod_{j \geq 1} \frac{1 - y_j t}{1 - y_j q t} = \prod_{j \geq 1} \frac{1 - y_j \cdot \frac{1}{q} \cdot (q t)}{1 - y_j \cdot (q t)} \\ &= \sum_{r \geq 0} q_r(Y; q^{-1}) q^r t^r, \end{aligned}$$

where $Y = \{y_1, y_2, \dots\}$. Comparing the coefficients from two sides we get the following corollary.

Corollary 3.11.

$$\begin{aligned} \rho_n &= q_n(Y; q^{-1}) q^n = (1 - q^{-1}) P_n(Y; q^{-1}) q^n \\ (3.10) \quad &= q^{n-1} (q - 1) P_n(Y; q^{-1}) = q^{n-1} (q - 1) \tilde{P}_n(Y; q). \end{aligned}$$

Corollary 3.11 gives the connection between the plethysm of the characteristic map of $\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}$ and the Hall-littlewood symmetric functions.

For any linear supercharacter [8, 7, 2] of U_n , there is a unique way to decompose the indexed set partition into connected components. For a linear supercharacter with k connected components, we can denote it by $\chi_{\vec{q}_1, \dots, \vec{q}_k}^{n_1 | n_2 | \dots | n_k}$ where for i from 1 to k , each n_i counts the size of the i^{th} connected component and $\vec{q}_i = (q_{i,1}, \dots, q_{i,n_i-1}) \in (\mathbb{F}_q^\times)^{n_i-1}$ denotes the labels of the arcs for the i^{th} connected component. The following corollary follows from the property of the linear supercharacters [8, 7, 2].

Corollary 3.12.

$$\rho \circ ch(\chi_{\vec{q}_1, \dots, \vec{q}_k}^{n_1 | n_2 | \dots | n_k} \uparrow_{U_n}^{G_n}) = \prod_{i=1}^k \rho_{n_i}.$$

Example 3.13. For the following linear supercharacter of U_6

$$\chi_{\vec{q}_1, \vec{q}_2, \vec{q}_3}^{1|2|3} = \chi \begin{array}{c} \bullet \quad \overset{q_{2,1}}{\curvearrowright} \bullet \quad \overset{q_{3,1} \ q_{3,2}}{\curvearrowright} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \end{array}$$

where $\vec{q}_1 = 0$, $\vec{q}_2 = (q_{2,1})$, $\vec{q}_3 = (q_{3,1}, q_{3,2})$ and $q_{2,1}, q_{3,1}, q_{3,2} \in \mathbb{F}_q^\times$, we have

$$\rho \circ ch(\chi_{\vec{q}_1, \vec{q}_2, \vec{q}_3}^{1|2|3} \uparrow_{U_6}^{G_6}) = \rho_1 \rho_2 \rho_3.$$

Let the transition matrix between $\{m_\lambda(X)\}_{\lambda \vdash n}$ and $\{p_\mu(X)\}_{\mu \vdash n}$ be $C_{\lambda, \mu}$, i.e.

$$m_\lambda(X) = \sum_{\mu} C_{\lambda, \mu} p_\mu(X).$$

Define $m_\lambda(q-1)$ by the following equation

$$m_\lambda(q-1) = \sum_{\mu} C_{\lambda, \mu} p_\mu(q-1),$$

where $p_n(q-1) = q^n - 1$ for every $n \geq 1$, and $p_\mu(q-1) = p_{\mu_1}(q-1) \cdots p_{\mu_l}(q-1)$ for $\mu = \{\mu_1, \dots, \mu_l\}$.

Remark 3.14. Using the orthogonal relation between the bases $\{m_\lambda\}$ and $\{h_\mu\}$, we give another expression for ρ_n as follows:

$$\rho_n = \sum_{\lambda \vdash n} m_\lambda(q-1) \cdot h_\lambda(Y).$$

Proof. Using the notation in Section 2.1, we have

$$\Omega(Yqt) = \prod_{j \geq 1} \frac{1}{1 - y_j qt}, \quad \Omega(-Yt) = \prod_{j \geq 1} (1 - y_j t).$$

$$\begin{aligned} CH(t) &= \prod_{j \geq 1} \frac{1 - y_j t}{1 - y_j qt} \\ &= \Omega[(q-1)Yt] \\ &= \sum_{n \geq 0} \left(\sum_{\lambda \vdash n} m_\lambda(q-1) \cdot h_\lambda(Y) \right) t^n. \end{aligned}$$

□

It seems that we lose much information by taking plethysms on the characteristic map of $\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}$. However if we only consider the induced characters from U_n to G_n , we can express the characteristic map of the induced characters in basis $\{\tilde{P}_\mu | \mu \in \mathcal{P}^\Phi\}$ from the results of doing plethysms. To show this fact, we first introduce the following homomorphism defined in [6]:

$$\omega : \Lambda_{\mathbb{C}}(Y) \rightarrow \Lambda_{\mathbb{C}}(Y)$$

by

$$\omega(e_r(Y)) = h_r(Y), \text{ for all } r \geq 0.$$

Lemma 3.15. ([6]) ω is an involution and automorphism on $\Lambda_{\mathbb{C}}(Y)$. Also, we have

$$\omega(p_r(Y)) = (-1)^{r-1} p_r(Y), \text{ for all } r \geq 0.$$

The following theorem illustrates the relation between the application of plethysms on the characteristic map in basis $\{S_\lambda | \lambda \in \mathcal{P}^\Theta\}$ and the characteristic map in basis $\{\tilde{P}_\mu | \mu \in \mathcal{P}^\Phi\}$.

Theorem 3.16. The following diagram commutes:

$$\begin{array}{ccc} \bigotimes_{\varphi \in \Theta} \Lambda_{\mathbb{C}}(Y^\varphi) & \xrightarrow{T} & \bigotimes_{f \in \Phi} \Lambda_{\mathbb{C}}(X_f) \\ \rho \downarrow & & \downarrow \Pi|_{\Lambda_{\mathbb{C}}(X_{f=x-1})} \\ \Lambda_{\mathbb{C}}(Y) & \xrightarrow{t \circ \omega} & \Lambda_{\mathbb{C}}(X_{x-1}) \end{array}$$

where T is the map of transformation from basis $\{S_\lambda | \lambda \in \mathcal{P}^\Theta\}$ to basis $\{\tilde{P}_\mu | \mu \in \mathcal{P}^\Phi\}$, t is the map of changing variables y_i into $X_{i,x-1}$ for $i = 1, 2, \dots$, and $\Pi|_{\Lambda_{\mathbb{C}}(X_{f=x-1})}$ is the projection to the space $\Lambda_{\mathbb{C}}(X_{x-1})$.

Proof. We rewrite equation (2.4) as follows

$$p_k(Y^\varphi) = (-1)^{n-1} \sum_{x \in M_n} \xi(x) p_{n/d(f_x)}(X_{f_x}),$$

where $\xi \in \varphi$, $x \in f_x$ and $n = k \cdot d(\varphi)$. If we apply plethysm on $p_k(Y^\varphi)$ we get $p_n(Y)$. Applying the projection map $\Pi|_{\Lambda_{\mathbb{C}}(X_{f=x-1})}$ on the right-hand side of equation (2.4) yields $(-1)^{n-1} p_n(X_{x-1})$. Since $\{p_n : n = 1, 2, \dots\}$ are algebraically independent over \mathbb{C} and $\{p_\lambda : \lambda \text{ a partition}\}$ form a basis for $\Lambda_{\mathbb{C}}$, we get the theorem from Lemma 3.15. \square

Corollary 3.17. *If we use $ch(\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n})(X_f : f \in \Phi)$ to denote the expression of the characteristic map of $ch(\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n})$ in terms of basis $\{\tilde{P}_\mu | \mu \in \mathcal{P}^\Phi\}$, then we have the following identity:*

$$\begin{aligned} ch(\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n})(X_f : f \in \Phi) &= t \circ \omega(\rho_n) \\ &= q^{n-1}(q-1)\omega(\tilde{P}_n(X_{x-1})). \end{aligned}$$

Proof. From the definition of the induced character by equation (2.5) we know that

$$\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}(g) = 0$$

for all $g \in G_n$ which are not similar to any unipotent upper-triangular matrices. Notice that the characteristic polynomial for all matrices in U_n is $(x-1)^n$. Since similar matrices have the same characteristic polynomial, $\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}$ could possibly take nonzero values only on those matrices in G_n with characteristic polynomials equal to $(x-1)^n$. We then have

$$ch(\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n})(X_f : f \in \Phi) \in \Lambda_{\mathbb{C}}(X_{x-1})$$

and so

$$\Pi|_{\Lambda_{\mathbb{C}}(X_{f=x-1})}[ch(\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n})] = ch(\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}).$$

By theorem 3.16 we obtain the corollary. \square

Remark 3.18. From the proof of Corollary 3.17 we conclude that for any character χ of U_n , if we induce χ from U_n to G_n , then we have

$$ch(\chi \uparrow_{U_n}^{G_n})(X_f : f \in \Phi) = t \circ \omega \circ \rho(ch(\chi \uparrow_{U_n}^{G_n})(Y^\varphi : \varphi \in \Theta)).$$

For $\lambda = \lambda_1, \dots, \lambda_l$ let $\rho_\lambda = \rho_{\lambda_1} \rho_{\lambda_2} \dots \rho_{\lambda_l}$. By Corollary 3.11 since $\rho_n = q^{n-1}(q-1)P_n(Y; q^{-1})$ we know that $\{\rho_\lambda\}$ forms a basis for the symmetric function ring $\Lambda_{\mathbb{C}}(Y)$. Thus $\rho(ch(\chi \uparrow_{U_n}^{G_n}))$ can be written into $\rho(ch(\chi \uparrow_{U_n}^{G_n})) = \sum_{\lambda \vdash n} C_\lambda \rho_\lambda$ where $C_\lambda \in \mathbb{C}$. We then define a map as follows.

Definition 3.19. Define $\hat{\rho} : \Lambda_{\mathbb{C}}(Y) \rightarrow \mathbb{C}\text{-span}\{S_\lambda | \lambda \in \mathcal{P}^\Theta\}$ by

$$\begin{aligned} \hat{\rho}(\rho_n) &:= \sum_{\substack{\lambda \in \mathcal{P}_n^\Theta \\ ht(\lambda)=1}} S_\lambda \\ &= ch(\chi_{(q_1, \dots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}). \end{aligned}$$

and

$$\hat{\rho}(\rho_\lambda) = \hat{\rho}(\rho_{\lambda_1})\hat{\rho}(\rho_{\lambda_2}) \cdots \hat{\rho}(\rho_{\lambda_l}),$$

where $\lambda = (\lambda_1, \dots, \lambda_l)$.

Proposition 3.20. *For a fixed finite field \mathbb{F}_q and a character χ of U_n , we have*

$$(\hat{\rho} \circ \rho)(ch(\chi \uparrow_{U_n}^{G_n})) = ch(\chi \uparrow_{U_n}^{G_n}).$$

Proof. Since ω is an automorphism and ρ is multiplicative, the proposition follows from Theorem 3.16 and Remark 3.18. \square

Suppose $\rho(ch(\chi \uparrow_{U_n}^{G_n})) = \sum_{\lambda \vdash n} C_\lambda \rho_\lambda$ where $C_\lambda \in \mathbb{C}$, from the definition of $\hat{\rho}$ we get

$$\begin{aligned} \hat{\rho} \circ \rho(ch(\chi \uparrow_{U_n}^{G_n})) &= \sum_{\lambda \vdash n} C_\lambda (\hat{\rho}(\rho_\lambda)) \\ (3.11) \qquad \qquad \qquad &= \sum_{\lambda \vdash n} C_\lambda \hat{\rho}(\rho_{\lambda_1}) \hat{\rho}(\rho_{\lambda_2}) \cdots \hat{\rho}(\rho_{\lambda_l}). \end{aligned}$$

Using Proposition 3.20 we get the following corollary.

Corollary 3.21. *For a fixed finite field \mathbb{F}_q and a character χ of U_n , suppose $ch(\chi \uparrow_{U_n}^{G_n}) = \sum_{\lambda \vdash n} C_\lambda \rho_\lambda$ where $C_\lambda \in \mathbb{C}$. We have*

$$\begin{aligned} ch(\chi \uparrow_{U_n}^{G_n}) &= \sum_{\lambda \vdash n} C_\lambda \rho_\lambda \\ &= \sum_{\lambda \vdash n} C_\lambda \left(\sum_{\substack{\lambda^{(1)} \in \mathcal{P}_{\lambda_1}^\Theta \\ ht(\lambda^{(1)})=1}} S_{\lambda^{(1)}} \right) \left(\sum_{\substack{\lambda^{(2)} \in \mathcal{P}_{\lambda_2}^\Theta \\ ht(\lambda^{(2)})=1}} S_{\lambda^{(2)}} \right) \cdots \left(\sum_{\substack{\lambda^{(l)} \in \mathcal{P}_{\lambda_l}^\Theta \\ ht(\lambda^{(l)})=1}} S_{\lambda^{(l)}} \right). \end{aligned}$$

Remark 3.22. It is difficult to get an expression for $ch(\chi \uparrow_{U_n}^{G_n})$ in terms of basis $\{S_\lambda | \lambda \in \mathcal{P}^\Theta\}$, which gives the irreducible decomposition of the induced character. However if we know the plethysm of the characteristic map of $\chi \uparrow_{U_n}^{G_n}$, we may use $\hat{\rho}$ to get the irreducible decomposition of $ch(\chi \uparrow_{U_n}^{G_n})$. We hope the results could contribute to research in this problem and we list some open problems in Section 4.

4. FURTHER QUESTIONS

The induced characters that we are studying in this paper are very special, so a natural question to ask is if we can give a nice formula for the characteristics of all the induced supercharacters from U_n to G_n . Zelevinsky [12] and Thiem and Vinroot [9] have worked on the case of degenerate Gelfand-Graev characters. The question of how the generalized Gelfand-Graev representations of the finite unitary group decompose is still open. The generalized Gelfand-Graev representations, which are defined by Kawanaka [5], are obtained by inducing certain irreducible representations from a unipotent subgroup [9]. Here the supercharacters that we are considering are more general than the case of the generalized Gelfand-Graev representations. We hope that the ideas and results developed in this paper could help to work on this direction.

Let us compute plethysms of the characteristic map of some induced supercharacters.

Example 4.1. For $q = 2$, we have

$$\begin{aligned} \rho \circ ch \left(\chi \begin{array}{c} \text{1} \\ \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \\ \uparrow_{U_3}^{G_3} \end{array} \right) &= (\rho_3 + \rho_2 \rho_1)|_{q=2} \\ \rho \circ ch \left(\chi \begin{array}{c} \text{1} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \quad 4 \\ \uparrow_{U_4}^{G_4} \end{array} \right) &= (\rho_4 + 2\rho_3 \rho_1 + \rho_2 \rho_1^2)|_{q=2} \\ \rho \circ ch \left(\chi \begin{array}{c} \text{1} \quad \text{1} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \quad 4 \\ \uparrow_{U_4}^{G_4} \end{array} \right) &= \rho \circ ch \left(\chi \begin{array}{c} \text{1} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \quad 4 \\ \uparrow_{U_4}^{G_4} \end{array} \right) = (2\rho_4 + \rho_2 \rho_2 + \rho_3 \rho_1)|_{q=2} \\ \rho \circ ch \left(\chi \begin{array}{c} \text{1} \quad \text{1} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \quad 4 \\ \uparrow_{U_4}^{G_4} \end{array} \right) &= (\rho_4 + \rho_3 \rho_1)|_{q=2} \end{aligned}$$

Inspired from these results, we give the following conjecture and open questions.

Conjecture 4.2. For a fixed finite field \mathbb{F}_q and a supercharacter χ of U_n , we have

$$\rho \circ ch(\chi \uparrow_{U_n}^{G_n}) \in \mathbb{N}[\rho_1, \dots, \rho_n].$$

If the above conjecture is true, then the following remark is meaningful.

Remark 4.3. For a fixed finite field \mathbb{F}_q and a character χ of U_n , suppose $ch(\chi \uparrow_{U_n}^{G_n}) = \sum_{\lambda \vdash n} C_\lambda \rho_\lambda$ where $C_\lambda \in \mathbb{C}$. We have

$$(4.1) \quad \dim(\chi) = \sum_{\lambda \vdash n} C_\lambda.$$

Proof. From Corollary 3.12 we have

$$\chi \uparrow_{U_n}^{G_n} = \sum_{\lambda \vdash n} C_\lambda (\chi_{\vec{q}_1, \dots, \vec{q}_l}^{\lambda_1 | \lambda_2 | \dots | \lambda_l} \uparrow_{U_n}^{G_n}) = \left(\sum_{\lambda \vdash n} C_\lambda \chi_{\vec{q}_1, \dots, \vec{q}_l}^{\lambda_1 | \lambda_2 | \dots | \lambda_l} \right) \uparrow_{U_n}^{G_n},$$

where $\vec{q}_i = (q_{i,1}, \dots, q_{i,\lambda_i-1}) \in (\mathbb{F}_q^\times)^{\lambda_i-1}$. So we have

$$\dim(\chi) = \dim \left(\sum_{\lambda \vdash n} C_\lambda \chi_{\vec{q}_1, \dots, \vec{q}_l}^{\lambda_1 | \lambda_2 | \dots | \lambda_l} \right) = \sum_{\lambda \vdash n} C_\lambda \dim(\chi_{\vec{q}_1, \dots, \vec{q}_l}^{\lambda_1 | \lambda_2 | \dots | \lambda_l}).$$

Since $\dim(\chi_{\vec{q}_1, \dots, \vec{q}_l}^{\lambda_1 | \lambda_2 | \dots | \lambda_l}) = 1$, we prove the remark. \square

Question 4.4. For a fixed finite field \mathbb{F}_q and a supercharacter χ of U_n , try to find a formula for the plethysms of the characteristic map of $\chi \uparrow_{U_n}^{G_n}$.

$$\rho \circ ch(\chi \uparrow_{U_n}^{G_n}) = \sum_{\lambda \vdash n} C_\lambda \rho_\lambda,$$

where $\rho_\lambda = \rho_{\lambda_1} \rho_{\lambda_2} \dots \rho_{\lambda_l}$ for $\lambda = \lambda_1, \dots, \lambda_l$. It is nice to give a combinatorial formula for the coefficient C_λ since the example above suggest a few possible rules.

Remark 4.5. If we have the formula of $\rho \circ ch(\chi \uparrow_{U_n}^{G_n})$, we can easily get the expression for the characteristic map of $\chi \uparrow_{U_n}^{G_n}$ in terms of basis $\{\tilde{P}_\mu | \mu \in \mathcal{P}^\Phi\}$ by Remark 3.18. We may also use $\hat{\rho}$ to get an expression in the basis $\{S_\lambda | \lambda \in \mathcal{P}^\Theta\}$ by Corollary 3.21.

Question 4.6. Up to now the induced representations that we are considering are in characteristic zero. Another problem we can think about is what happens in characteristic p case.

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